Generalised symmetries of Fokker-Planck-type equations

1989 J. Phys. A: Math. Gen. 22 L453

(http://iopscience.iop.org/0305-4470/22/11/001)

J. Phys. A: Math. Gen. 22 (1989) L453-L456. Printed in the UK

LETTER TO THE EDITOR

Generalised symmetries of Fokker–Planck-type equations

G Cicogna† and D Vitali‡

† Dipartimento di Fisica, Università di Pisa, Piazza Torricelli 2, 56100-Pisa, Italy
‡ Scuola Normale Superiore, 56100-Pisa, Italy

Received 7 February 1989

Abstract. We study extended symmetry properties of time-evolution partial differential equations, including Fokker–Planck-type equations. We show that, even if—in general—no non-trivial symmetry is present, in some particular interesting cases some special symmetry is allowed, and we provide the general method for finding it.

Considerable interest has recently been devoted to the determination of the 'generalised symmetries' of a differential equation. Referring to [1] for a full exposition of this topic, let us recall briefly here that, if, for example, a time-evolution partial differential equation has a solution $f = f(x, t)$, $x \in R$, $t \in R$, a generalised symmetry is any continuous transformation (possibly non-linear or only local) $x \to x'$, $t \to t'$, $f \to f'$ such that $f'(x', t')$ is also a solution of the given equation. These transformations are assumed to depend analytically on a real parameter ε, so that attention is mainly centred on their Lie generators, which in this case can be written in the general form

$$v = \xi(x, t, f)\frac{\partial}{\partial x} + \tau(x, t, f)\frac{\partial}{\partial t} + \phi(x, t, f)\frac{\partial}{\partial f} \tag{1}$$

where ξ, τ, ϕ are the functions to be determined.

The generalised symmetries of the heat equation, of the wave equation, of the Korteweg-de Vries equation, among others, are listed in [1]; an important case concerning the Schrödinger equation is examined in [2]. Some mathematical aspects of this problem are discussed in [1, 3]. In this letter, we investigate the symmetry properties of one-dimensional equations of Fokker–Planck type [4, 5], namely equations of the form

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(a(x)f) + \frac{1}{2}\frac{\partial^2}{2\partial x^2}(g^2(x)f) \tag{2}$$

or also in the more general form

$$\frac{\partial f}{\partial t} = A(x)f + B(x)\frac{\partial f}{\partial x} + C(x)\frac{\partial^2 f}{\partial x^2} \tag{3}$$

where a, g, or respectively A, B, C are given regular (analytical) functions of $x \in R$, with

$$C(x) = \tfrac{1}{2}g^2(x) \neq 0. \tag{3'}$$

Following step by step the Olver procedure in [1], we construct the second prolongation of the vector field (1) and apply it to equations (2) or (3) in order to find the conditions on the functions ξ, τ, ϕ in such a way that (1) generates a symmetry of equations (2) and (3). The conditions we obtain are the following (we refer to equation (3), the subscripts meaning differentiations):

$$\tau_x = \tau_f = 0 \qquad \xi_f = 0 \qquad \phi_{ff} = 0 \tag{4}$$

implying $\tau = \tau(t)$, $\xi = \xi(x, t)$ only, and

$$\phi = \alpha(x, t) + \beta(x, t) f \tag{5}$$

(this is identical to the case of the heat equation [1]), and

$$\xi_x = \xi \frac{C_x}{2C} + \frac{\tau_t}{2}. \tag{6}$$

This equation can be solved with respect to x, to get

$$\xi = c(t) g + \tfrac{1}{2}\tau_t g G \tag{7}$$

where $c = c(t)$ is a function of t and $G = G(x)$ is an integral function of $1/g(x)$. The two other conditions we find are

$$B_x \xi + B(\tau_t - \xi_x) + \xi_t + C(2\phi_{xf} - \xi_{xx}) = 0 \tag{8}$$

$$A_x \xi f + A(\tau_t - \phi_f) + A\phi + B\phi_x + C\phi_{xx} - \phi_t = 0. \tag{9}$$

Now using (5), we see that (9) requires that $\alpha = \alpha(x, t)$ be a solution of (3); then we can rewrite the two above conditions (8) and (9) in the form

$$2C\beta_x = B\xi_x - B_x\xi - \xi_t - B\tau_t + C\xi_{xx} \tag{10}$$

$$\beta_t = A_x \xi + B\beta_x + C\beta_{xx} + A\tau_t. \tag{11}$$

Finally, using (7), these can be written

$$\beta_x = p_1 c + p_2 c_t + p_3 \tau_t + p_4 \tau_{tt} \qquad p_i = p_i(x) \tag{12}$$

$$\beta_t = q_1 c + q_2 c_t + q_3 \tau_t + q_4 \tau_{tt} \qquad q_i = q_i(x) \tag{13}$$

where

$$p_1 = \frac{Bg_x}{g^2} - \frac{B_x}{g} + \frac{g_{xx}}{2} \qquad p_2 = -\frac{1}{g}$$
$$p_3 = \frac{Bg_x G}{2g^2} - \frac{B}{2g^2} - \frac{B_x G}{2g} + \frac{g_{xx} G}{4} + \frac{g_x}{4g} \qquad p_4 = \frac{-G}{2g} \tag{12'}$$

and

$$q_1 = Bp_1 + \frac{g^2 p_{1x}}{2} + A_x g, \qquad q_2 = Bp_2 + \frac{g^2 p_{2x}}{2}$$
$$q_3 = Bp_3 + \frac{g^2 p_{3x}}{2} + \frac{A_x g G}{2} + A, \qquad q_4 = Bp_4 + \frac{g^2 p_{4x}}{2}. \tag{13'}$$

In order to find β from (12) and (13), we have now to impose the condition $\beta_{tx} = \beta_{xt}$, which becomes, observing that $p_1 = q_{2x}$ and $p_3 = q_{4x}$, and using also (12')

$$c_{tt} + g q_{1x} c = -(\tfrac{1}{2} G \tau_{ttt} + g q_{3x} \tau_t). \tag{14}$$

All quantities in (14) depending on x are determined by the initial equation (3); so, imposing the identity (14) amounts to giving some restrictions on the functions $c(t)$, $\tau(t)$. Once one has found the most general c, τ consistent with (14), one can evaluate $\beta(x, t)$ through (12) and (13), and then obtain, via equations (1), (5) and (7), the complete symmetry of any equation of the type (2) and (3), as the foregoing examples will illustrate.

First of all, it is clear that, whatever the functions a, g, A, B, C (and therefore p_i, q_i in (13)) are, the above system of equations possesses the solution

$$\tau = c_1 \qquad \beta = c_2 \qquad \phi = \alpha + c_2 f \qquad c = 0 \qquad \xi = 0 \tag{15}$$

(where c_1, c_2 are constants and α a solution of (2) and (3)), which corresponds to the symmetries generated by

$$\boldsymbol{v}_1 = \frac{\partial}{\partial t} \qquad \boldsymbol{v}_2 = f\frac{\partial}{\partial f} \qquad \boldsymbol{v}_3 = \alpha\frac{\partial}{\partial f}. \tag{16}$$

Clearly, these are 'trivial' symmetries: in fact, $\boldsymbol{v}_1$ expresses the time-translation invariance (equations (2) and (3) are indeed autonomous equations); $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ are a consequence, as is well known [1], of the linearity of (2) and (3) (if f and α are solutions of (2) and (3), the same is true for $kf + \alpha$).

Inspecting now more closely condition (14), one easily sees that 'generically', i.e. if there is no special relationship between the functions G, gq_{1x}, gq_{3x} appearing in (14), the only solution allowed by (14) is just $c = 0$, $\tau_t = 0$, which leads precisely to the 'trivial' situation (16) shared by all autonomous linear equations, as stated above. More precisely, it is not difficult to conclude that if the functions G, gq_{1x}, gq_{3x}, 1 are linearly independent, the only allowed symmetry is just the trivial one. For instance, this is the case if one chooses

$$a = 1 \qquad g = x \tag{17}$$

in equation (2).

Some non-trivial symmetries can arise if some relationship occurs between the functions of x in (14), as shown by the foregoing examples. Assume, for example, in (2)

$$a = x \qquad g = 1 \tag{18}$$

which corresponds of course to an important case in the Fokker-Planck theory [4, 5], then condition (14) becomes

$$c_{tt} - c = -\tfrac{1}{2}x(\tau_{ttt} - 4\tau_t)$$

which implies

$$c_{tt} = c, \qquad \tau_{ttt} = 4\tau_t.$$

Once c, τ, β are evaluated, one gets that, with this choice, the new symmetries generated by

$$\boldsymbol{v}_4 = \mathrm{e}^{t}\frac{\partial}{\partial x} \qquad \boldsymbol{v}_5 = \mathrm{e}^{-t}\frac{\partial}{\partial x} + 2x\,\mathrm{e}^{-t}f\frac{\partial}{\partial f}$$

$$\boldsymbol{v}_6 = x\,\mathrm{e}^{2t}\frac{\partial}{\partial x} + \mathrm{e}^{2t}\frac{\partial}{\partial t} - \mathrm{e}^{2t}f\frac{\partial}{\partial f} \qquad \boldsymbol{v}_7 = x\,\mathrm{e}^{-2t}\frac{\partial}{\partial x} - \mathrm{e}^{-2t}\frac{\partial}{\partial t} + 2x^2\,\mathrm{e}^{-2t}f\frac{\partial}{\partial f}$$

are present. Now, v_4 expresses the property that if $f(x, t)$ solves the equation, then also $f(x+\varepsilon\, e^t, t)$, $\varepsilon \in R$, does; v_5 states also that

$$\exp(2\varepsilon x\, e^{-t} + \varepsilon^2 e^{-2t}) f(x+\varepsilon\, e^{-t}, t)$$

is a solution, whereas the remaining two operators generate more complicated symmetries involving simultaneously x, t, f.

Another interesting case for the Fokker–Planck equation is

$$a = x \qquad g = x \tag{19}$$

for which (14) gives

$$c_{tt} = \tau_{ttt} = 0$$

leading to the symmetries generated by

$$v_4 = x\frac{\partial}{\partial x} \qquad v_5 = xt\frac{\partial}{\partial x} - \left(\ln x + \frac{t}{2}\right) f\frac{\partial}{\partial f}$$

$$v_6 = \frac{x}{2}\ln x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{1}{4}\left(\ln x + \frac{t}{2}\right) f\frac{\partial}{\partial f}$$

$$v_7 = xt\ln x\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - \frac{1}{2}\left(t\ln x + \ln^2 x + t + \frac{t^2}{4}\right) f\frac{\partial}{\partial f}.$$

Note that here v_4 expresses the rather elementary property of scale invariance $x \to e^\varepsilon x$ of the equation in this case.

It can be remarked finally that with the choice

$$a = x \qquad g = x + 1 \tag{20}$$

only trivial symmetries (16) survive. We do not consider here the symmetries of the case $a = 0$, $g = 1$, i.e. the heat equation, which are obviously included (with $c_{tt} = \tau_{ttt} = 0$) in the above general scheme, being fully discussed in [1].

References

[1] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)

[2] Rudra P 1987 *Phys. Rev.* A **36** 5

[3] Ovsjannikov L V 1962 *Group Properties of Differential Equations* (Novosibirsk) (Engl. transl. 1967 by G Bluman, Vancouver, BC)

[4] van Kampen N G 1981 *Stochastic Processes in Physics and Chemistry* (Amsterdam: North-Holland)

[5] Haken H 1983 *Advanced Synergetics* (Berlin: Springer)